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A Study on Primary School Students' Arrangement and Transformation of Structures and Arithmetic Representations of the Marble Arrangement Problem

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The purpose of this study was to explore the richness and transformations of arithmetic representation and arrangement of structure of students through the lens of the marble arrangement problem. The participants in the study were 12 Grade 5 students producing a total of 144 responses during two stages of problem solving. The researcher collected worksheets and class interaction videos with individual student's explanations. The correctness of the responses was checked according to structure arrangement approaches and arithmetic representation from students' work. Quantitative statistics of work were reported to show richness of outcomes, and found that the students' performance types were concentrated in logical operations and topological operations, totaling 81.3%. Qualitative analysis was on transformation of students' approaches from Stage 1 to Stage 2 by referring to teacher-student interactions and only to students who exhibited a change. The findings showed that: first, the three most commonly used methods were topological operations, logical operations, and algebraic operations; and second, students demonstrated three transformation paths: (a) promoting unitization by dividing objects into equal

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parts using algebraic and logical operations in structures, (b) expanding operational ability through mapping of iterating and numbers, and (c) refinement of arithmetic generalization through generating of unitization and operation.

Keywords: arrangement structure; arithmetic representation; marble arrangement problem; transformation

Introduction

Patterning was often seen as a useful way to express mathematical thinking, and recent research of interdisciplinary interest has placed special emphasis on the identification and application of patterns. Determining whether students can successfully understand structure arrangement of objects were worth exploring in mathematics education studies. The process of patterning is driven by the possibility of purposely organizing representations of the external world, in other words, by the search of structures.

In order to understand students' shape structures and problem-solving performance, Silver et al. (1995) provided a marble arrangement problem, in which students counted marbles when arrangement of 25 marbles were given in a specific way. This paper-and-pencil task of "finding as many ways as you can" required students to observe and see structure. Students exhibited a variety of modes of explanation and solution approaches and made shifts in approaches. However, the reasons why students used these strategies were not explored. In this study, through the lens provided by the marble arrangement problem, we focused on how students solved this problem and interacted in a classroom environment, and also on students' employing algebraic operations, logical operations, topological operations, iterating, and generating to arrange the structure of objects before presenting the results in arithmetic expressions. This study was guided by two research questions:

1. After teaching, how did Grade 5 students interact in the classroom and show their arrangement of marbles and arithmetic representations?
2. How did the thinking transformation take place from Stage 1 to Stage 2, from students' arrangement of marbles and arithmetic representations?

Literature Review

Dynamic Infrastructure of Mind

Owing to the specific ontological nature of mathematical objects as abstract and mostly relational entities, the construction of meanings as mental models involves the construction of new mental objects and relationships (Steinbring, 2005). This construction must be in line with acquisition of new linguistic, graphical, and symbolic means for expressing them (Schleppegrell, 2010). By transitioning between verbal, symbolic, graphical, and concrete representations, students can construct the mental objects and relations to which a mathematical concept refers. In consequence, connecting representations is established as a fruitful teaching strategy. Dynamics seems to be a constant characteristic of the human mind and human species. Such mechanisms are involved in the dynamic infrastructure of mind (DIM). Singer (2009) identified seven operational clusters (Table 1). These have been denominated based on their major components as follows: associating, comparing, algebraic operations, logical operations, topological operations, iterating, and generating. A short description of each category is given below:

Table 1: Singer's (2009) Seven Operational Clusters

Categories of operations	Associating	Comparing	Algebraic operations	Logical operations	Topological operations	Iterating	Generating
Some basic elements	- Recognizing - Naming - Reproducing - Representing - Classifying - Isomorphic transformations	- Estimating - Selecting - Discriminating - Checking - Numerical comparison	- Proto quantitative operations - Operations with sets - Arithmetical operations - Operations with variables	- Using logical operators - Using quantifiers	- Identifying boundaries - Identifying limits - Identifying convergences	- Mimicking - Identifying patterns - Developing recurrences	- Grasping - Guessing - Conditioned generating
Targets	- Building equivalent metaphors	- Building cross metaphor systems and metonymies	- Operating with discrete quantities - Digital approach	- Constructing meta-systems intermediated by language	- Operating with continuity - Analogical approach	- Developing recursive processes	- Developing intrinsic motivation
Basic connections and symmetries	Contextual experience		Duality of matter/processes			Emergence	

1. *Associating* — Including operations described as connecting two entities based on a one-to-one correspondence. The capacity of building one-to-one correspondences evolves from its primitive form of one-to-one matching objects, to associating one-to-one various representations. It also favors, through symmetry, building the roots of analogical reasoning;
2. *Comparing* — Containing operations described as connecting an entity to one or more others, based on a relationship;
3. *Algebraic operations* — As inner operations in the algebraic cluster, the proto-quantitative operations or pre-arithmetical operations refer to putting together, taking away, magnifying, reducing, adding, splitting, combining, sharing, folding, and others that, quantitatively expressed, lead to addition, subtraction, multiplication, division, squaring, and so on;
4. *Logical operations*— Referring to the capacity to use basic connectors (conjunction, disjunction, negation, quantifiers) as main composites for combining actions, or propositions;
5. *Topological operations* — Referring to identify boundaries, relate them with discrete components, perceive objects globally, and cross the frontier between discrete and continuous. The primitive topological property of mind leads students to globally perceive continuous surfaces (Feigenson et al., 2002);
6. *Iterating* — Based on the recursive capacity of mind, recursion is fundamental for survival because it allows automatize and economize knowledge and skills. Iterating is an essential component in trial-and-error mechanisms;
7. *Generating* — Described as an operational category, the elements of it create new entities, previously unknown, starting from entities already known. A special element in this category is grasping, which allows perceiving an entity or its essence instantaneously, without proceeding discursively in space or time.

We obtained inspirations from Singer’s (2009) DIM theory:

1. The development of students’ mathematical concepts is the result of the interaction of multiple abilities, including the application of language, skills, symbols, and specific representations;
2. Students’ concepts will become more refined and proficient with age and the challenges they face in the environment;
3. When faced with new situations or problems, students will use existing prototypes to adapt and solve problems.

This study was to explore the richness and transformations of arithmetic representation and arrangement of structure of students through the lens of the marble arrangement problem. The focus was on the above operational clusters. In particular, students' performance was classified based on algebraic operations, logical operations, topological operations, iterating, and generating.

Arrangement of Structure

Arrangement of structure is defined as the process of contemplating the shape and arrangement in space, such as the deformation of objects and the movement of objects and other entities in space (Duval, 2017; Hegarty, 2010; Mulligan et al., 2020; Silver et al., 1995). Therefore, the arrangement includes the recognition of structures and positions of shapes and the application of problem-solving strategies. Silver et al. (1995) explored the performance of students in paper-and-pencil assignments on the marble arrangement problem, identifying the following framework of strategies applied by the students:

1. *Enumeration* provides evidence of some object-counting process, such as counting one after another, counting in a specific direction, or counting by drawing a continuous line;
2. *Find-a-structure* uses equal division, which involves placing the same number of marbles in each group, or forms groups according to some convenient arrangement such as rows, columns, diagonals, or any combination of these;
3. *Change-the-structure* rearranges the marbles through displacement, drawing arrows to show their new positions or adding (subtracting) additional marbles to facilitate the calculation process.

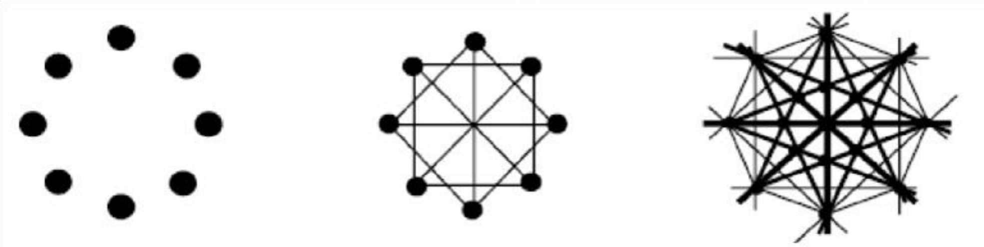
Thus, the following types of aggregate structures might be differentiated as distinct theoretical entities:

1. Rigid structure characterized by: (a) oversized, very stable nuclei, (b) a poorly developed network, sometimes totally lacking, and (c) associations that function in the area of the recognition of a standard situation and its reproduction;
2. Flexible structure characterized by: (a) stable nuclei, (b) a developed network, and (c) associations based on recognizing invariant elements in various environments. A flexible structure allows problem solving through analogy and inductive or deductive inferences when the context is relatively familiar;
3. Dynamic structure characterized by: (a) flexible nuclei that are or could become structures in their turn, (b) complex networks with ramifications and hierarchies, and (c)

dynamic associations that facilitate quick mobilization of the structure through the discovery of critical paths.

These associations stimulate the self-development of the structure, highlight underlying relations among different structures, and give rise to links between various structures within the cognitive system (Singer, 2009). In Figure 1, there are representational schemes for the three types of structures. The schemes highlight the 3 dimensions used to emphasize the differences among those types: nuclei, network, potential associations beyond the network. Thus, while in a rigid structure, the nuclei are very developed; in a flexible structure, they diminish in favor of the network. This process continues for a dynamic structure, in which the connections become the most important part, capable to engage new nuclei and to extend beyond the existing structure.

Figure 1: Singer's (2009) Three Aggregate Structures



A rigid structure | A flexible structure | A dynamic structure

From this review, the framework by Silver et al. (1995) allowed us to understand students' strategies; Singer's (2009) approaches assisted us to understand students' thinking as they aggregate structures while added possible explanations to students' solutions.

Methods

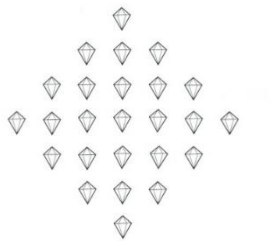
Participants

The research sample came from a Grade 5 class in an elementary school, with 6 boys and 6 girls each joining an after-school creativity club. In addition to the same national language and mathematics subjects as in ordinary classes, the curriculum of the class includes subjects related to the curriculum benchmarks of the Ministry of Education's (2014) standard art class in the flexible curriculum. These benchmarks included exploration, understanding relationships, appreciation, discover connections, and carrying out observations. In order to meet the above objectives, this study provided the marble arrangement problem as task. Before trying this task, students already learned multiplication, length measurement, basic concepts of shapes and their properties, two-step unions, and arithmetic operations on integers, factors, and multiples. Members of the club were having common interests in visual arts, performed well and were interested in mathematics.

Task design and administration

First, we provided a worksheet based on the marble arrangement problem for the students to work on individually. The worksheet contained the following instructions: "A-Bao and his classmates went on a treasure hunt and found a pile of marbles in a cave. They arranged them as shown in Figure 2. How many marbles are there in total?"

Figure 2: The Marble Arrangement Problem



Next, the teacher encouraged students by using prompts: (a) "Think about it first. How can we count the number of marbles?" (b) "Try to break down the shape into smaller parts and color them. This will help you find the expression used to calculate this quantity." (c) "Can you transform this unusual number into one that everyone knows?"

Later, two stages of investigations followed:

1. *Stage 1* (40 minutes) — The teacher provided the students with six illustrations of the marble arrangement problem (in a fixed order: work from left to right and from top to bottom), and asked them to observe the characteristics of the given objects using the following prompt: "What arrangement structure do you make? What else do you make?" The teacher then asked the students to create novel designs and explore new possibilities, and encouraged them to share their results.
2. *Stage 2* (40 minutes) — In this stage, the six illustrations are provided again and students were asked to once again work from left to right and from top to bottom. Since

they already knew there were a total of 25 marbles, the teacher asked "Is it possible to draw other shapes?" to encourage students to reflect on their own work to produce ideas in ways different from those given in Stage 1. Students also shared their results to the group.

Data collection and analysis

The twelve participants (S1 to S12) produced a total of 144 arrangement structures. Data included (a) worksheets and (b) class interaction video. When presenting their work, students were asked to sequentially describe their creative experiences and explain why certain key actions were taken. The teacher asked questions such as "Where did you start?", "What was your first thought about this?", and "Why did you make changes on these arrangement structures?". Each presentation was recorded on video and transcribed.

Silver et al.'s (1995) previous study on students' marble arrangement problems provided the basis for methods and performance types. However, their study was to use paper-and-pencil tests to analyze students' performance. The purpose of this study was to explore students' arrangement structures and arithmetic representation of the marble arrangement problem and their transformations, hoping to observe from students' tasks how they know the number of objects in a collection by counting or dividing it. The development rules are suggested through alternation of some graphic symbols displayed one after another (line and circle, etc.), hoping that they can organize a system which highlights the sequential generation of its subcomponents by repetition at different scales. Finally, they can construct different structures based on their cognitive development as they face task challenges. In order to more deeply explore students' performance in the two stages of homework and the discourse focus of classroom interaction output, and understand how their concepts change, Singer's (2009) DIM framework can provide the method for this study's analysis. Therefore, we define the types of analysis methods (see Table 2).

For quantitative data analysis, the correctness of the responses was checked according to arrangement structures and then arithmetic representations.

The framework of analysis was divided into three categories of rigid, flexible, and dynamic, and then subdivided into five approaches (I to V). These five approaches were operation clusters from Table 1. Approach I resulted from rigid structures: through counting objects by arbitrary selection, then summing them up with calculations. For approach II, students also used rigid structures, but they did an orderly arrangement of objects (from left to right or from top to bottom) to divide the image into strips by lines. Approaches I and II

Table 2: Characteristics of Three Categories and Five Approaches of Arrangement Structure

Category/Approach	Characteristic
1. Rigid structure	
I. Algebraic operations	• Enumerate marbles by arbitrary selection.
II. Logical operations	• Use lines to circle or concatenate marbles to form identical units.
2. Flexible structure	
III. Topological operations	• Group objects into identical blocks, then add the remaining marbles; or combine several blocks.
3. Dynamic structure	
IV. Iterating	• After grouping objects into several different units, then add the remaining marble, and count using the same units.
V. Generating	• Use equal groups (the same objects) combined into identical units, then deducted the number of overlapping marbles to create new unitization.

are similar to Singer's (2009) algebraic operations and logical operations. Because the student knows that the number of marbles is 25, association and comparison are not processed. Students using approach III adopted the flexible structure strategy. Approach III was to group objects into identical blocks, then add the number of marbles or combine blocks. This approach is similar to Singer's topological operations. Finally, approaches IV and V used the dynamic structure strategy. Approach IV divided the shapes with different object quantities, and the remaining objects were summed up. When the shapes were divided and when an object was repeated, students deducted after summing up. Approach V used a divided shape and produced a certain part of the figure. The two approaches are like Singer's iterating and generating. Students used equal groups (the same objects) and combined objects into identical units, then deducting the number of overlapping marbles, to create new unitization. After coding, descriptive statistics were used to report on the richness of approaches to solve the problem regarding arrangement structure and arithmetic representations.

As for qualitative analysis (on transformation), the investigators interpreted the teacher-student interactions transcribed from the videos taken during both stages of the task implementation. Two independent investigators read the transcriptions and the way students reported on processes was studied. The thought of arrangement structures was identified according to Singer (2009) and thinking approaches such as rigid, flexible and dynamic structures were used to analyze paths of students' transformation from arrangement structures to arithmetic representations. Inconsistencies were handled to ensure that internal reliability was reached. Finally, the two investigators decided on transformation paths of thinking from Stage 1 to Stage 2, which only included those students that exhibited a change in Stage 2.

Results and Discussion

Arrangement Structures and Arithmetic Representations

Data coding was done using the framework as in Table 1. Table 3 presents statistics related to these approaches applied by the students to the marble arrangement problem. The three most commonly used methods were flexible structure (topological operations) (approach III, 54.2%), rigid structure (logical operations) (approach II, 27.1%), and rigid structure (algebraic operations) (approach I, 11.1%), totaling to 92.4%. Each approach divided the 25 objects and arranged them into equal blocks of the same unit or the same shape, which was convenient for multiplication. However, by unitizing the shape, some objects cannot be included in the blocks, so additional points were required. The arithmetic representations shown by students demonstrated their structure.

Table 3: Visuospatial Categories/Approaches Applied by Students in Stage 1 and Stage 2

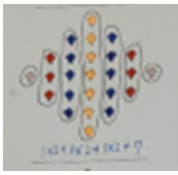
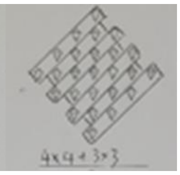
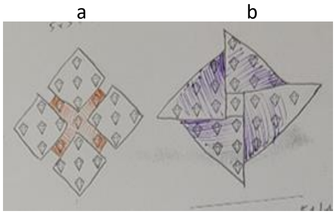
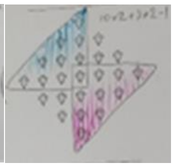
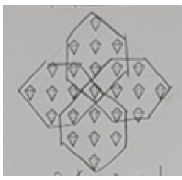
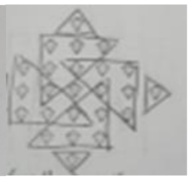
Category/Approach	Total (%) *N* = 144	Stage 1 (%) *N* = 72	Stage 2 (%) *N* = 72
1. Rigid structure			
I. Algebraic operations	16 (11.1)	8 (11.1)	8 (11.1)
II. Logical operations	39 (27.1)	21 (29.2)	18 (25.0)
2. Flexible structure			
III. Topological operations	78 (54.2)	39 (54.2)	39 (54.2)
3. Dynamic structure			
IV. Iterating	4 (2.8)	2 (2.8)	2 (2.8)
V. Generating	7 (4.9)	2 (2.8)	5 (6.9)

In both stages, the most commonly used methods were the same three: flexible structure (topological operations) (54.2%, 54.2%), rigid structure (logical operations) (29.2%, 25.0%), and rigid structure (algebraic operations) (11.1%, 11.1%). In Stage 2, there was a preference for topological operations over logical operations or algebraic operations. This preference in Stage 2 (after producing 6 approaches in Stage 1) represented the discursion attention described by Singer's (2009) flexible structure strategy. Also found in Stage 2, dynamic structure was applied more often (9.7% > 5.6%), having 7 students (only 4 students in Stage 1) representing the structure-change strategy described by Silver et al. (1995).

The proportion of topological operations belonging to flexible structure was very high. From this, it is presumed that students had the marbles divided, organized, and unitized. The object "unitization" was used as a basis for creation and categorization, generalization (integrating parts into a whole), positioned/drawn in a regular or predictable manner (e.g., via rotating and duplicating the pattern).

Table 4 shows the five approaches of analysis based on the arrangement structures and arithmetic representations presented by the students' assignments. The table included actual examples of students' work in each of the five approaches.

Table 4: Arrangement Structure Categories/Approaches and Related Arithmetic Representations

Category/Approach	Identification of shape structures		Arithmetic representation
1. *Rigid structure* I. Algebraic operations	a 	b 	a: 5 + 4 + 4 + 6 + 6 = 25 b: 2 × 2 + 4 + 5 + 6 + 1 × 6 = 25
1. *Rigid structure* II. Logical operations	a 	b 	a: 1 + 3 + 5 + 7 + 5 + 3 + 1 = 25 b: (4 × 4) + (3 × 3) = 25
2. *Flexible structure* III. Topological operations	a 	b	a: (5 × 4) + 5 = 25 b: 6 × 4 + 1 = 25
3. *Dynamic structure* IV. Iterating	a 	b 	a: 3 × 4 + 3 × 4 + 1 = 25 b: 10 × 2 + 3 × 2 – 1 = 25
3. *Dynamic structure* V. Generating	a 	b 	a: 8 × 4 – 4 – 4 + 1 = 25 b: 6 × 4 – 4 + 5 = 25

We applied qualitative analysis to the collected data, and the 144 arrangement responses (72 from each of the two stages) were coded as approaches I, II, III, IV, V according to Table 2. Six students who changed the approach in Stage 2 were further studied, to answer the second research question on transformation.

Transformation of Arithmetic Representations and Arrangement Structures Applied by Students

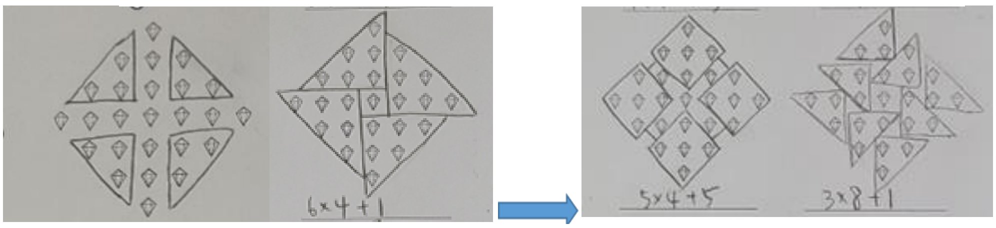
There were 6 students who exhibited a change in Stage 2. Obvious differences in students' arithmetic representations and in arrangement structures (named as transformation later on) were found. A total of three transformation paths will be reported below: the unitization of objects, the mapping of visuospatial thinking and numbers, and the generalization of arithmetic representations.

Path 1 (S2, S3): Promoting unitization by dividing objects into equal parts using algebraic and logical operations in structures

Unitization referred to the regular partition of objects within a given spatial distribution, forming like diagrams to facilitate multiplication. It indicated a qualitative change in arrangement structures. Most students successfully counted objects in the marble arrangement problem using algebraic and logical operations by unitizing the objects and dividing them into equal parts. Below were two students who conducted unitization (S2, S3). For example, see Figure 2 for thinking of S2.

Figure 2: Arrangement Structures of S2's Thinking From Stage 1 to Stage 2

2(a) 2(b) 2(c) 2(d)



Stage 1 Stage 2

T: How did you calculate the number of marbles? [Teacher asking about Stage 1]

S2: I saw a cross in the structure. There are triangles formed by marbles all around it. After the triangle is circled, it looks like 2(a). The number of marbles can then be calculated like this: 3 × 4 + 7 + 6 = 25. 7 is the number of marbles of the [vertical] line in the middle, and 6 is 3 + 3 on both sides. In 2(b), I saw a point in the middle of the cross in 2(a), so I add the top left triangle in 2(a) to the 3 marbles above the middle line on its right. The point then becomes a triangle with 6 points on the upper left as in 2(b). In this way, 4 triangles can be circled, and there is only 1 point left in the middle. Using 6 × 4 + 1 = 25, the total number of marbles can also be calculated as 25.

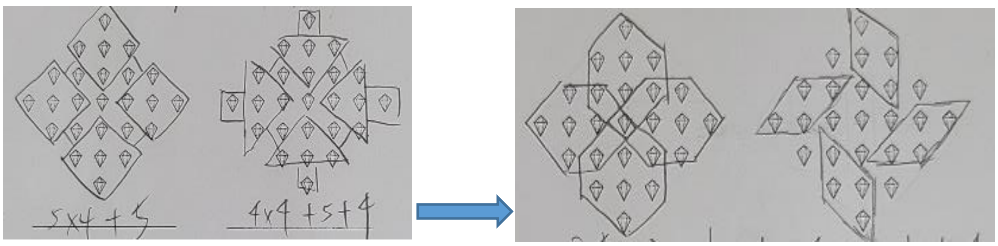
T: What about 2(c) and 2(d)? [Teacher directing attention to Stage 2]

S2: The cross in 2(a) is viewed upright. After I rotated it 45 degrees, the cross looked like the one in 2(c). I then circled the points on the four sides of the cross to form a square. There are 5 points in each square, so 5 × 4 + 5 = 25, since the cross has 5 points. In 2(d), I wanted to design a rotating toy windmill; because there are 25 points, I left a center point as the axis. For the remaining 24 points, they are grouped into eight groups of three. Starting from the top of the shapes, I circle the first right triangle, continuing in a clockwise manner until 2(d) is completed. The result is 3 × 8 + 1 = 25.

After S2 gave explanations, the teacher talked to S3 (see Figure 3 for thinking of S3).

Figure 3: Arrangement Structures of S3's Thinking From Stage 1 to Stage 2

3(a) 3(b) 3(c) 3(d)



Stage 1 Stage 2

T: How did you calculate the total number of marbles?

S3: I did it the same way as S2. I first saw a cross-shaped structure in the task, with the same five marbles at the top, bottom, left, and right. I circled it and drew four squares like 3(a), which gave me 5 × 4 + 5 = 25; in 3(b), I changed the squares in 3(a) into arrows, with four arrowheads and one tail. The result is now 4 × 4 + 5 + 4 = 25.

T: Then what about 3(c) and 3(d)? (Attention to Stage 2)

S3: From the squares in 3(a) (there are five marbles in all), I combined one marble on each of the four sides of the cross and the one in the middle (three marbles in all), forming four overlapping hexagons, each with six sides. There are eight marbles in each hexagon, making for a total of $8 \times 4 = 32$ marbles. After deducting the middle one which was counted four times and the three from overlapping structures, I get $8 \times 4 - 1 \times 4 - 3 = 25$. In 3(d), the five marbles in the intersecting line segment of 3(c) are combined, and the four outer marbles are grouped within a parallelogram. This structure is similar to the windmill of S2, and the total number of marbles can be calculated as $5 + 4 \times 4 + 4 = 25$.

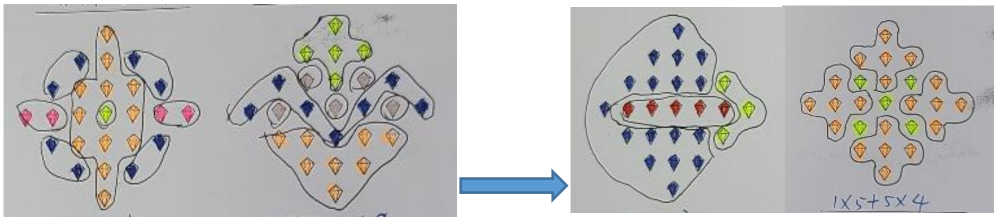
To compare, in Stage 1, the two students (S2 and S3) used features such as crosses and the center point as the basis for their thinking. They then divided the remaining objects into equal parts, grouping them into the same arrangement structures, and used multiplication to obtain their final results. In Stage 2, S2 switched to the construction of a structure dividing the marbles into equal parts. S2 used the real-world object of a windmill as the inspiration for the design, defining every three points as a unit after deducting the point on the axis. Eight right triangles were drawn in a clockwise manner, giving a sense of rotation. S3 used the structure created in Stage 1, incorporating the structures of the windmill, integrating the original number of objects into the number of objects in the cross, and forming a new unit before deducting the number of overlapping objects in the structure. The researchers considered students' unitized performance as structural ability found in Silver et al. (1995): using an equal division approach, involving placing the same number of marbles in each arrangement structure, or forming groups according to some convenient permutation. They also followed the approaches of arrangement structures proposed by Singer (2009). Students recognized features in the structures before they further used operations, unitized objects, then used multiplication for arithmetic generalization.

Path 2 (S4, S6): Mapping of iterating and numbers with operational ability expanded

The skills of iterating and combination not only promoted the unitization of objects in arrangement structures, but also helped students to construct and complete thinking and to unitize digital mapping. For example, S4 and S6 completed their design through the organization of objects. For example, see Figure 4 for thinking of S4.

Figure 4: Arrangement Structures of S4's Thinking From Stage 1 to Stage 2

(4a) 4(b) 4(c) 4(d)



Stage 1 Stage 2

T: 4(a) and 4(b) look like … animals. What animals are they? How did you do it?

S4: In 4(a) I drew a sea turtle. It has a square-shaped shell on its belly (3 × 3), with a head and tail (2 × 2) and two fins (2 × 2). Together with the shell on its back (2 × 4), we have a total of 3 × 3 + 2 × 2 + 2 × 2 + 2 × 4 = 25 marbles. 4(b) is a prince with a hat. There are two feathers on it (7 marbles) and a triangular headscarf (8 marbles). He has bright eyes and mouth (3 marbles), a cross on his head (5 marbles) and two gems (2 marbles), giving a total of 7 + 8 + 3 + 5 + 2 = 25.

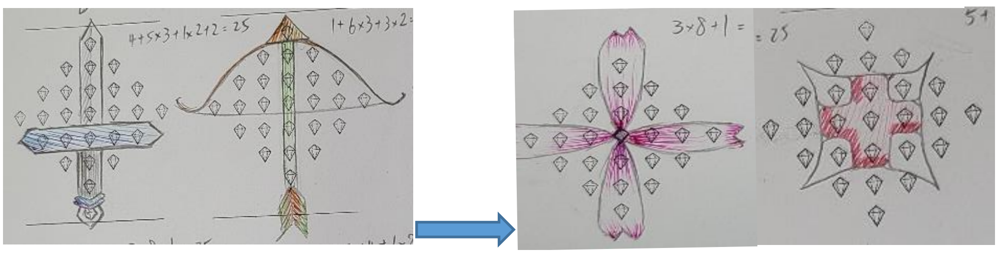
T: What about 4(c) and 4(d)? (Attention to Stage 2)

S4: In 4(c), I transformed the shapes into a flying bird. From the top, one can see that it has a head, a body, and wings on both sides. The total number of marbles needed is 8 × 2 + 3 + 5 + 1 = 25. Inspiration for 4(d) came from the cross on the prince's head in 3(b). Since there are five marbles in each cross, 5 × 5 = 25.

Another example was from student S6. Please see Figure 5.

Figure 5: Arrangement Structures of S6's Thinking From Stage 1 to Stage 2

(5a) 5(b) 5(c) 5(d)



Stage 1 Stage 2

T: In 5(a) and 5(b), you drew a sword and bow and arrow set. Why are 5(c) and 5(d) different?

S6: At the beginning, I saw a cross with marbles in the middle of the shapes, so I thought of it as a sword and a bow and arrow. Although I knew that 25 marbles were used, I could not write down the formula. The one I listed was a mess; there were no shapes in the way I counted. Later, I mainly focused on the center point of the cross. There are three points on the top, bottom, left, and right of this point. I circled these three points to draw a flower braid, and since there were three flower buds between the flower braids like in 5(c), there are a total of 8 such shapes, resulting in 3 × 8 = 24. After adding the marble in the middle, the result is 25, which makes a neat formula. In 5(d), I circled the cross in the middle, and then discovered that there is a single marble on all four sides, giving a total of four. Together with the cross, one can draw a square-shaped region that has four marbles each on the left, right, top, and bottom, resulting in equation 4 × 5 + 5 = 25. Not only is it pretty, but I can also quickly determine how many marbles there are.

From the above dialogues, we can see that in Stage 1, the number of circled objects varies, resulting in difficulties regarding the counting and arithmetic representation. In Stage 2, the objects were manipulated based on the characteristics of the structures. The structures were thus divided into equal parts and grouped to form units with the same number of objects. This unitization promotes the use of multiplication and the generalization of arithmetic.

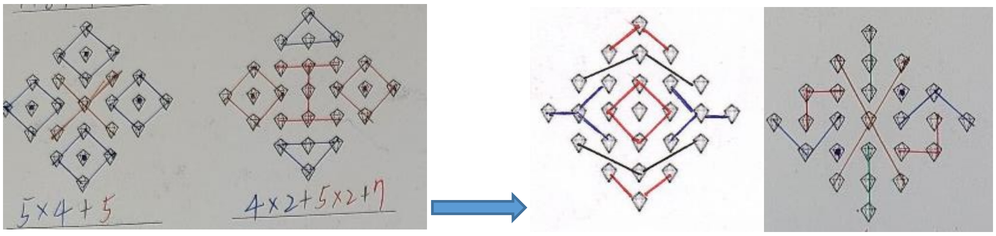
Students individually and sequentially identified each element in the structures as they performed the count, matching the elements in the sequence in which each element was spoken in the same order throughout to reveal the numerical value. Through the coordination of the two different representations of the marking unit, the name and the arithmetic formula of the combined number, the two representations were mapped one-to-one with their respective meaning units, and finally, the arrangement structures and structures formed by thinking were presented.

Path 3 (S8, S11): Refinement of arithmetic generalization through generating of unitization and operation

In addition to using generating to unitize objects and facilitate the presentation of arithmetic representations, students S8 and S11 unitized objects and then divided and generated them to simplify their operations, promoting arithmetic generalization (See Figure 6).

Figure 6: Arrangement Structures of S8's Thinking From Stage 1 to Stage 2

6(a) 6(b) 6(c) 6(d)



Stage 1 Stage 2

T: How did you draw 6(a) and 6(b)? (Referring to Stage 1)

S8: I first connected the marbles in the middle with red lines (five in all). There are five marbles at the left, right, top, and bottom of the resulting cross. Using blue lines, I drew four squares, with the result of 5 × 4 + 5 = 25. In 6(b), I changed the crossed lines into an "H" shape, so the blue squares became triangles (with four marbles each). The result was then 4 × 2 + 5 × 2 + 7 = 25.

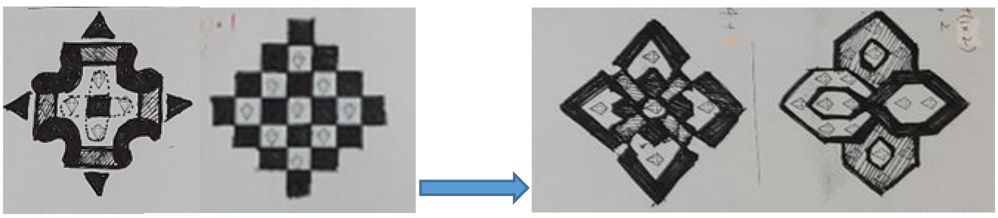
T: What about 6(c) and 6(d)? (Referring to Stage 2)

S8: In 6(a) and 6(b), I grouped together 3, 4, and 5 marbles. Therefore, I transformed 6(c), connecting every three marbles into a string. For the top and the bottom, there are a total of four strings; for the left and right, there are a total of two strings with four marbles each. Together with the five marbles in the middle, there are a total of 3 × 4 + 4 × 2 + 5 = 25 marbles. In 6(d), I connected three marbles into a string with two for the top and the bottom (connected with green lines) and two each for the left and the right (connected using red and blue lines, respectively). Combined with the large cross in the middle and two extra marbles on the sides (marked in black), the equation became 3 × 6 + 5 + 2 = 25. These shapes are beautiful and symmetric; don't you think the shapes I drew are nice?

The next student is S11 and the thinking was given in Figure 7.

Figure 7: Arrangement Structures of S11's Thinking From Stage 1 to Stage 2

7(a) 7(b) 7(c) 7(d)



Stage 1 Stage 2

T: Your creation is unique, how did you come up with it?

S11: I like to paint in black and white, so after looking at the marble problem, I colored the center point black first. Its left, right, top, and bottom appear white (1 marble each, 4 in all), so I colored its left, right, top, and bottom points gray (also 1 marble each, 4 in all). I then colored the corner black to form a curved shape (4 bars with 3 marbles each) and also colored the upper, lower, left, and right marble on the outermost layer black, creating 7(a). $4 \times 3 + 3 \times 4 + 1 = 25$ can be used to calculate the number of marbles. 7(b) is drawn using black and white inlays; there are 4 black lines and 3 white lines, so $4 \times 4 + 3 \times 3 = 25$. 7(c) is actually created by moving the curved black line of 7(a) to the periphery. Black marbles become white and white marbles become black. The number of marbles can be calculated as $(3 \times 4) + (4 \times 3) + 1 = 25$. In 7(d), I turned the graphic into one with the same left/right and top/bottom sections. There are four marbles in the upper and lower shapes, surrounding a single one. The left and right shapes are made up of five marbles, with two each inside them. There are another two marbles in the inner circle. The formula is $4 \times 2 + 1 \times 2 + 5 \times 2 + 2 + 2 + 1 = 25$.

The interviews with S8 and S11 indicated that their designs were based on their creative habits; that is, using lines to connect objects in series to form units and using colors to distinguish and organize objects to form units. Not only were the creations of the two students unique, but they were also able to generalize their counting methods using arithmetic and explained in detail how they perceived the arrangement structures. For example, S11 used parentheses in the formula, illustrating the close connection between arithmetic representations.

From the above results, students spontaneously looked for arrangement structures and relationships and would try to notice and generate symmetric shapes. Dynamic views included transformations like reflections, translations and rotations, and corresponding symmetries in dynamic structures to collaborate to facilitate generalization of the formula as noted in Singer (2009). Students' operations and unitization strategies and methods were found. Perhaps this was due to the topological operations produced by the marble arrangement problem, or it may be the result of the interaction between students' thought processes. Regardless of the results, the arrangement structures presented by students in this study were not linear nor unidirectional, but was intertwined, requiring the use of thought and problem-solving abilities.

We summarized three thought transformation paths:

1. *Operation* → *Unitization* — For example, S4, S6, and S11 mainly used arrangement structures, observing the number of objects in the structures, then dividing and iterating

the original structures. This formed topological operations, and the number of objects after unitization was mapped to produce a numerical representation.

2. *Unitization → operation* — For example, S3 and S8 used algebraic operations and logical operations setting the same number of objects as a unit, mapping it into an arrangement structure, then dividing and combining the objects in the original structures to produce mathematical representations.
3. *Interaction* — Students S2 and S3 in Stage 2 interacted with operations and unitization. They utilized algebraic operations, logical operations, topological operations, iterating, and generating of the original objects to produce more complex structures. They conducted more sophisticated mapping of numbers to symbols, facilitating refinement of arithmetic generalization.

To move from arrangement structures to the output of arithmetic representations, students need to perceive the objects and their relationships. Students can then apply these perceptions to divide spatial objects into equal parts, and unitize them to allow for the mapping of symbols and numbers. The thinking of arrangement structures can be transformed into arithmetic representations. In the marble arrangement problem, object elements such as center points, crosses, and intersecting line segments can be perceived. A relatively simple operation skill such as equalization or combination can then be applied, followed by linear methods such as "operation → unitization" or "unitization → operation" to map symbols and generalize arithmetic. However, in Stage 2, when students already found multiple ways to solve the problem, they were able to use unitization and operation to produce more exceptional works, echoing the operation clusters described by Singer (2009), and the structure-change strategy described by Silver et al. (1995). Students also used higher-level manipulation skills such as dynamic structures; and explained the resulting arithmetic representations using the thought skills involved in topological operations, iterating, and generating.

Very few studies have described the features of shape structures and math-related learning tasks (Schoevers et al., 2020). In this study, the marble arrangement problem successfully stimulated students' thought processes of arrangement structures and arithmetic representations. This marble arrangement problem was unique in enabling students to do exploration and generalization. The "simple counting of 25 marbles" was also expressed as rows addition ($4 \times 4 + 3 \times 3$) and or simply moving one marble to each corner with five rows of five (5×5). However, Kawaguchi (1961) remarked that when extending the arrangement to 5 marbles on each side (not 4), the total number was not a square number anymore and the pattern could not be generalized. Thus, this problem is unique. Our results reminded us that the creation of novel designs requires solid perception, manipulation, and unitization skills. Skills such as algebraic operations, logical operations, topological operations, iterating, and generating, as well as the mapping between objects and symbols need to be strengthened in primary-school geometry courses. Further, students may also be encouraged to try different methods and solve a problem again and again.

Conclusions

The purpose of this study was to explore the arithmetic representations and arrangement structures of Grade 5 students through the lens of the marble arrangement problem. We sought to determine how different arrangement structure approaches have resulted in a thought transformation. The approaches applied to the marble arrangement problem were: (a) algebraic operations, (b) logical operations, (c) topological operations, (d) iterating, and (e) generating with unitization. The three most commonly used methods were: find-a-structure (topological operations), enumeration (logical operations), and enumeration (algebraic operations). Although approaches used were about the same in the two stages, the proportions of approaches were significantly different. In the course of creation, students demonstrated the following thought transformations: (a) promoting unitization by dividing objects into equal parts using features in structures, (b) mapping of visuospatial thinking and numbers with operational ability expanded, and (c) refinement of arithmetic generalization through collaboration of unitization and operation.

The findings of this study highlight the close connection between the development of arrangement structure and mathematical concepts (Silver et al., 1995; Singer, 2009). These underline the importance of mastering various arrangement structures. Students should thus be encouraged to construct drawings and measurements to participate in the modeling, characterization, arrangement structures, and generalization of object regularities. This study found that students will use algebraic operations, logical operations, topological operations, iterating, and generating to divide objects into equal parts to promote the function of unitization, expand the ability of an operation to expand the mapping of spatial thinking and numbers, and cooperate with unitization and operation to promote the improvement of arithmetic generalization. However, the participants were elite students in arts. They were included as a convenient sample who took extra lessons after school and do not represent ordinary students. Although this is a limitation, their performances suggest activities for geometry or creation-related education (Ministry of Education, 2014; Mulligan et al., 2020).

In addition, the researchers only collected and analyzed the data regarding arrangement structures and thought transformation through worksheets and video. Technologies, such as eye tracking and gesture tracking, could provide more objective and refined results.

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國小學生彈珠結構排列變換與算術表徵的研究

陳嘉皇、梁淑坤

摘 要

本研究旨在透過彈珠排列問題來探討學生算術表徵和結構排列的豐富性和轉化。研究參與者是 12 名五年級學生，他們在問題解決的兩個階段中總共做出了 144 個答案。研究人員收集學生的作業表單和課堂互動視頻，給出個別學生的解釋，並根據學生作業的排列結構方法和算術表徵來檢查答案的正確性。學生作業的量化統計顯示出豐富的成果，發現學生的表現類型集中在邏輯運算和拓樸運算，兩者共佔了 81.3%。質性分析透過參考師生互動，並且只針對表現出變化的學生，來研究學生從第一階段到第二階段的方法轉變。結果發現：首先，最常用的三種方法是拓樸運算、邏輯運算和代數運算；其次，學生展示了三種轉換路徑：（1）透過使用結構中的代數和邏輯運算將物件分成相等的部分來促進單元化，（2）迭代和數字的對應擴展了運算能力，（3）透過組成單元和運算來精煉算術推演。

關鍵詞：結構排列；算術表徵；彈珠排列問題；變換

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